

THE DEFECT OF FANO 3-FOLDS

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ABSTRACT. This paper studies the rank of the divisor class group of terminal Gorenstein Fano 3-folds. If Y is not \mathbb{Q} -factorial, there is a small modification of Y with a second extremal ray; Cutkosky, following Mori, gave an explicit geometric description of the contractions of extremal rays on terminal Gorenstein 3-folds. I introduce the category of weak-star Fanos, which allows one to run the Minimal Model Program (MMP) in the category of Gorenstein weak Fano 3-folds. If Y does not contain a plane, the rank of its divisor class group can be bounded by running a MMP on a weak-star Fano small modification of Y . These methods yield more precise bounds on the rank of $\mathrm{Cl} Y$ depending on the Weil divisors lying on Y . I then study in detail quartic 3-folds that contain a plane and give a general bound on the rank of the divisor class group of quartic 3-folds. Finally, I indicate how to bound the rank of the divisor class group of higher genus terminal Gorenstein Fano 3-folds with Picard rank 1 that contain a plane.

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1. INTRODUCTION

Let $Y_4 \subset \mathbb{P}^4$ be a quartic hypersurface with terminal singularities. The Grothendieck–Lefschetz theorem states that Cartier divisors on Y are restrictions of Cartier divisors on \mathbb{P}^4 , i.e. that $\mathrm{Pic} Y \simeq \mathbb{Z}[\mathcal{O}_Y(1)]$. However, no such result holds for $\mathrm{Cl} Y$, the group of Weil divisors of Y . If the quartic Y is not factorial, very little is understood about some aspects of its topology.

More generally, a Fano 3-fold Y with terminal Gorenstein singularities is a 1-parameter flat deformation of a nonsingular Fano 3-fold with

the same Picard rank [Nam97]: these are classified in [Isk77, Isk78, MM86]. However, as in the case of terminal quartic 3-folds, the rank of $\mathrm{Cl} Y$ is not known in general.

A normal projective variety is \mathbb{Q} -factorial if an integral multiple of every Weil divisor is Cartier. A terminal Gorenstein Fano 3-fold Y is \mathbb{Q} -factorial if and only if it is factorial [Kaw88, Lemma 6.3], that is, if and only if

$$H^2(Y, \mathbb{Z}) \simeq H_4(Y, \mathbb{Z}).$$

Factoriality is a global topological property: it depends both on the analytic type of singularities and on their relative position. If $Y_4 \subset \mathbb{P}^4$ is a nodal quartic hypersurface with at most 8 singular points, then Y is factorial [Che06]. Whereas the presence of a small number of ordinary double points does not affect the factoriality of a hypersurface, this is not true for even slightly more complicated types of singularities. For instance, a quartic Y_4 in the linear system Σ spanned by the monomials

$$\{x_0^4, x_1^4, (x_4^2 x_3 + x_2^3) x_0, x_3^3 x_1, x_4^2 x_1^2\}$$

on \mathbb{P}^4 is not factorial because the plane $\Pi = \{x_0 = x_1 = 0\}$ lies on Y . Yet, the general element $Y_4 \in \Sigma$ has a unique cA_1 singular point $P = (0:0:0:0:1)$ [Mel04].

In this paper, I study the rank of the divisor class group, or equivalently the *defect*, of terminal Gorenstein Fano 3-folds with Picard rank 1. The defect of a terminal Gorenstein Fano 3-fold Y is defined as the difference:

$$\sigma(Y) = \mathrm{rank} \mathrm{Cl} Y - \mathrm{rank} \mathrm{Pic} Y = h_4(Y) - h^2(Y).$$

The defect is a global topological invariant that measures how far Y is from being factorial, or, in other words, to what extent Poincaré duality fails on Y .

The notion of defect was introduced in [Cle83] in an attempt, based on Deligne's Mixed Hodge Theory [Del74], to extend Griffiths' results [Gri69] to the Hodge theory of double covers of \mathbb{P}^3 branched in a nodal hypersurface. Works on the defect of hypersurfaces or of Fano 3-folds have focussed on studying their mixed Hodge structures [Cle83, Dim90, NS95, Cyn01]. These results rely on computations of the cohomology of specific varieties: they are impractical to determine a global bound on the defect of terminal Gorenstein Fano 3-folds. In this paper, I use birational geometry to bound the defect of terminal Gorenstein Fano 3-folds. These methods also yield more precise bounds on the defect in terms of the Weil divisors lying on the 3-fold.

A nodal quartic hypersurface in \mathbb{P}^4 has at most 45 ordinary double points [Var83]; the Burkhardt quartic (Example 4.3) is the unique such

quartic up to projective equivalence[dJSBVdV90]. One sees that the defect of the Burkhardt quartic is at least 15. I show that the defect of any quartic $Y_4 \subset \mathbb{P}^4$ with terminal singularities is at most 15 and that this bound can be refined in some cases. More precisely, I prove:

Theorem 1.1. *Let $Y_4 \subset \mathbb{P}^4$ be a quartic hypersurface with terminal singularities. The rank of $\mathrm{Cl} Y$ satisfies:*

- (i) $\mathrm{rank} \mathrm{Cl} Y \leq 9$ when Y contains neither a plane nor a quadric,
- (ii) $\mathrm{rank} \mathrm{Cl} Y \leq 10$ when Y does not contain a plane,
- (iii) $\mathrm{rank} \mathrm{Cl} Y \leq 16$, with equality precisely when Y is projectively equivalent to the Burkhardt quartic.

Outline. I now sketch the proof of Theorem 1.1 and present the organisation of the paper.

In Section 2, I define the category of *weak-star Fano 3-folds*. A small factorization of a terminal Gorenstein Fano 3-fold Y is weak-star Fano unless Y contains a plane embedded by $|A_Y|_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$. Weak-star Fano 3-folds are inductively Gorenstein: if $\varphi: X \rightarrow X'$ is an extremal birational contraction, then X' has terminal Gorenstein singularities. This is a major simplification: one may then use the explicit geometric description of extremal contractions of Mori–Cutkosky [Cut88].

In Section 3, I show that more is true: the MMP may be run in the category of weak-star Fano 3-folds. Since weak-star Fano 3-folds are small modifications of terminal Gorenstein Fano 3-folds, some of their numerical invariants—such as their anticanonical degree—can be read off the classification of nonsingular Fano 3-folds [Isk77, Isk78, MM82]. Following the description of [Cut88], it is possible to associate to each extremal contraction some numerical constraints: this reduces many questions on the MMP of weak-star Fano 3-folds to a careful analysis of the classification [Isk77, Isk78, MM82].

Let Y be a terminal Gorenstein Fano 3-fold and X a small factorization; note that $\mathrm{rank} \mathrm{Cl} Y = \mathrm{rank} \mathrm{Pic} X = \rho(X)$. The main case is when X is a weak-star Fano 3-fold of Fano index 1. The rank of $\mathrm{Cl} Y$ is bounded by running the MMP on X and by studying the numerical constraints associated to divisorial contractions, especially those with centre along a curve. Roughly, the cases of high defect for Y correspond to the MMP on X consisting of the contraction of several low degree surfaces and ending with a high anticanonical degree Fano 3-fold, such as \mathbb{P}^3 (see Sections 3.1 and 3.2 for precise statements). The case of Fano index ≥ 2 is treated briefly in Section 3.3. Since $-K = 2H$ for H Cartier, a birational extremal contraction can only be a flop or the inverse of the blowup of a smooth point.

Section 4 is independent of the rest of the paper and treats the case of terminal Gorenstein Fano 3-folds whose small factorization is not weak-star. I study in detail quartic 3-folds that contain a plane and bound their defect by explicit calculation. I also indicate how to bound the defect of terminal Gorenstein Fano 3-folds with Picard rank 1 and genus $g \geq 3$.

A “geometric motivation” of non-factoriality. The framework of weak-star Fano 3-folds allows one to state a more precise description of the divisor class group of non-factorial Fano 3-folds [Kal]. If Y is not factorial, there is a small modification X of Y with a second extremal ray R ; in most cases, R is of divisorial type. Now, X can be deformed to a nonsingular small modification of a Fano 3-fold X_η with $\rho(X_\eta) = 2$, on which the extremal ray R is still present. It is then possible to run a two-ray game on X_η , and this two-ray game deforms back to a two-ray game on X . This shows that Y contains a surface of one of a finite number of special types (see [Kal]). This analysis can be carried out at each divisorial step of the MMP on a small factorization of Y , and can be used to exhibit generators for the Cox ring of Y .

Notations and conventions. All varieties considered in this paper are normal, projective and defined over \mathbb{C} . Let Y be a terminal Gorenstein Fano 3-fold, $A_Y = -K_Y$ denotes the anticanonical divisor of Y . The *Fano index* of Y is the maximal integer such that $A_Y = i(Y)H$ with H Cartier. As I only consider Fano 3-folds with terminal Gorenstein singularities in this paper, the term index always stands for Fano index. The *degree* of Y is H^3 and the *genus* of Y is $g(Y) = h^0(X, A_Y) - 2$. I denote by Y_{2g-2} for $2 \leq g \leq 10$ or $g = 12$ (resp. V_d for $1 \leq d \leq 5$) terminal Gorenstein Fano 3-folds of Picard rank 1, index 1 (resp. 2) and genus g (resp. degree d). Finally, $\mathbb{F}_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m))$ denotes the m th Hirzebruch surface.

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2. THE CATEGORY OF WEAK-STAR FANO 3-FOLDS

In this section, I recall some results on terminal Gorenstein Fano 3-folds and on the MMP for 3-folds. I also introduce the category of *weak-star Fano* 3-folds. Most terminal Gorenstein Fano 3-folds have a weak-star Fano small factorization. These are inductively Gorenstein and hence well behaved under the birational operations of the MMP.

2.1. Definitions and first results.

Definition 2.1.

1. A 3-fold Y with terminal Gorenstein singularities is *Fano* if its anticanonical divisor A_Y is ample.
2. A 3-fold X with terminal Gorenstein singularities is *weak Fano* if A_X is nef and big.
3. The morphism $h: X \rightarrow Y$ defined by $|nA_X|$ for $n \gg 0$ is the *(pluri-)anticanonical map* of X , $R = R(X, A)$ is the *anticanonical ring* of X and $Y = \text{Proj } R$ is the *anticanonical model* of X .
4. A weak Fano 3-fold X is a *weak-star Fano* if, in addition:
 - (i) A_X is ample outside of a finite set of curves, i.e. $h: X \rightarrow Y$ is a small modification,
 - (ii) X is factorial, and in particular X is Gorenstein,
 - (iii) X is inductively Gorenstein, that is $(A_X)^2 \cdot S > 1$ for every irreducible divisor S on X ,
 - (iv) $|A_X|$ is basepoint free, so that $\varphi_{|A|}$ is generically finite.

Remark 2.2.

1. Item (iii) in the definition of weak-star Fano 3-folds guarantees that if $\varphi: X \rightarrow X'$ is a birational extremal contraction, then X' is factorial and terminal. Indeed, by Lemma 2.5 and by Mori-Cutkosky's classification (see Theorem 2.6), X' is factorial and terminal unless φ contracts a plane $E \simeq \mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$ to a point of Gorenstein index 2. In this case, by adjunction, the restriction of A_X to E would be $A_{X|E} = \mathcal{O}_{\mathbb{P}^2}(1)$ and $(A_X)^2 \cdot E \leq 1$: this is impossible when X is weak-star.
2. Item (iv) of the definition is a convenience, that I have included to avoid digressing throughout the paper to exclude the special cases listed in Proposition 2.4. In addition, note that when $|A_X|$ is basepoint free, either $R(X, A_X)$ is generated in degree 1 and A_Y is very ample or $\varphi_{|A|}$ is 2-to-1 to \mathbb{P}^3 or to a quadric $Q \subset \mathbb{P}^4$.

Lemma 2.3. *Let $h: X \rightarrow Y$ be a small factorization of a terminal Gorenstein Fano 3-fold Y such that $|A_Y|$ is basepoint free.*

1. If A_Y is very ample, X is not weak-star Fano if and only if Y contains a plane \mathbb{P}^2 with $A_{Y|_{\mathbb{P}^2}} = \mathcal{O}_{\mathbb{P}^2}(1)$.
2. If A_Y is not very ample, X is not weak-star Fano if and only if $Y = Y_6 \subset \mathbb{P}(1^4, 3)$ and Y contains a plane $\{y = l(x_0, \dots, x_3) = 0\}$, where x_i are the coordinates of weight 1, y is the coordinate of weight 3 on $\mathbb{P}(1^4, 3)$ and l is a linear form.

Proof. Assume that A_Y is very ample. If Y contains a plane Π of the given form, the proper transform $S = h_*^{-1}\Pi$ of Π satisfies $A_X^2 \cdot S = 1$ and X is not weak-star Fano. Conversely, if X contains an irreducible divisor S with $(A_X)^2 \cdot S = 1$, the image of S is a surface in \mathbb{P}^{g+1} , and as $A_Y = \mathcal{O}_Y(1)$, and its degree in \mathbb{P}^{g+1} is 1. Similarly, if A_Y is not very ample, X fails to be weak-star Fano precisely when Y is a sextic double solid and contains a plane of the stated form. When $Y = Y(2, 4) \subset \mathbb{P}(1^5, 4)$ is a double cover of a quadric $Q \subset \mathbb{P}^4$, X is weak-star unless Y contains a plane of the form $\Pi = \{y = x_0 = x_1 = 0\}$, where x_i denote the coordinates of weight 1 and y is the coordinate of weight 4. However, in this case, Y has nonisolated singularities. \square

Proposition 2.4. *Let X be a weak Fano 3-fold. The anticanonical linear system $|A_X|$ is basepoint free except in the following cases:*

1. $\text{Bs } |A_X| \simeq \mathbb{P}^1$ is a curve lying on the nonsingular locus of X . The general member $S \in |A_X|$ is a nonsingular K3 surface and $\text{Bs } |A_X|$ is a -2 -curve on S , X is called monogonal and is one of:
 - (i) $X = \mathbb{P}^1 \times S$, where S is a weighted hypersurface of degree 6 in $\mathbb{P}(1^2, 2, 3)$; X has Picard rank 10,
 - (ii) $X = \text{Bl}_\Gamma V$, the blowup of a weighted hypersurface V of degree 6 in $\mathbb{P}(1^3, 2, 3)$ along the plane $\Pi = \{x_0 = x_1 = 0\}$; X has Picard rank 2.
2. $\text{Bs } |A_X| = \{p\}$ is an ordinary double point and Y is birational to a special complete intersection $X_{2,6} \subset \mathbb{P}(1^4, 2, 3)$ with a node; Y is the degeneration of a double sextic.

Proof. This result is well known and classical; I just sketch its proof (see [Kal07] for a complete proof). Let X be a weak Fano 3-fold; Reid [Rei83] shows that the general section S of $|A_X|$ is a K3 surface with no worse than rational double points. An easy extension of Saint-Donat's results [SD74] on nef and big linear systems on nonsingular K3 surfaces applied to $|A_X|_{|S}$ finishes the proof. \square

Lemma 2.5 (Cone theorem for weak Fano 3-folds). *Let X be a weak Fano 3-fold. The Kleiman–Mori cone $\text{NE}(X) = \overline{\text{NE}}(X)$ of X is a finite rational polyhedral cone. If $R \subset \text{NE}(X)$ is an extremal ray, then either*

$A_X \cdot R > 0$, or $A_X \cdot R = 0$ and there is an irreducible divisor D such that $D \cdot R < 0$. There is a contraction morphism φ_R associated to each extremal ray R . If φ_R is small, it is a flopping contraction.

Proof. The 3-fold X is terminal, hence by the standard cone theorem [KMM87, Theorem 4-2-1]:

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{A_X \leq 0} + \sum C_j$$

where the extremal rays C_j are discrete in the half space $\{A_X > 0\}$ and can be contracted. Since A_X is nef, for any $z \in N^1(X)$, $A_X \cdot z \leq 0$ if and only if $A_X \cdot z = 0$. The anticanonical divisor A_X is big: for some integer $m > 0$, $m(A_X) \sim A + D$, where A is an ample divisor and D is effective. By the Nakai–Moishezon criterion for ampleness, if $A_X \cdot z = 0$ then $D \cdot z < 0$. In particular, for $0 < \epsilon \ll 1$,

$$\overline{\text{NE}}(X) \subset -A_{X < 0} \cup (-A_X + \epsilon D)_{< 0}.$$

By the usual compactness argument, $\text{NE}(X)$ is a finitely generated rational polyhedral cone. Extremal rays can be contracted by the contraction theorem [KMM87, Theorem 3-2-1]. Finally, if ϕ_R is small, R flips or flops. A flipping curve γ on a terminal 3-fold X satisfies $A_X \cdot \gamma < 1$ [Ben85]. The anticanonical divisor is Cartier and nef: ϕ_R is a flopping contraction. \square

For clarity of exposition, I recall the classification of extremal divisorial contractions established by Cutkosky following Mori.

Theorem 2.6. [Cut88, Theorems 4 and 5] *Let X be a 3-fold with terminal factorial singularities and $f: X \rightarrow X'$ the birational contraction of an extremal ray. Assume that f is not an isomorphism in codimension 1. Then X' is \mathbb{Q} -factorial and:*

E1 If $f: X \rightarrow X'$ contracts a surface E to a curve Γ , X' is factorial, non-singular near Γ and f is the blowup of I_Γ . The curve Γ is l.c.i., has locally planar singularities and X has only cA_n singularities on E .

If $f: X \rightarrow X'$ contracts a surface E to a point p , f is one of the following:

- E2 $(E, \mathcal{O}_E(E)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$, f is the blowup at a non-singular point p .*
- E3 $(E, \mathcal{O}_E(E)) \simeq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1, -1))$, f is the blowup at an ordinary double point p .*
- E4 $(E, \mathcal{O}_E(E)) \simeq (Q, \mathcal{O}_Q(-1))$, where $n \geq 3$ and Q is a quadric cone in \mathbb{P}^3 , f is the blowup at a cA_{n-1} singular point p .*

E5 $(E, \mathcal{O}_E(E)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2))f$ is the blowup of a non Gorenstein point of index 2.

Theorem 2.7. [Nam97] *Let X be a small modification of a terminal Gorenstein Fano 3-fold. There is a 1-parameter flat deformation of X*

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \Delta \end{array}$$

such that the generic fibre \mathcal{X}_η is a nonsingular small modification of a terminal Gorenstein Fano 3-fold. The Picard ranks, the anticanonical degrees and the indices of X and \mathcal{X}_η are equal.

3. MMP FOR WEAK-STAR FANO 3-FOLDS

Let Y be a terminal Gorenstein Fano 3-fold, and let $h: X \rightarrow Y$ be a small factorization. In Section 2, I showed that unless Y contains a plane, X is weak-star. In this section, I show that the category of weak-star Fano 3-folds is preserved by the birational operations of the MMP, so that the explicit geometric description of extremal contractions of Mori–Cutkosky applies. Each type of extremal contraction imposes numerical constraints on the intersection numbers of relevant cycles in cohomology. Given that numerical invariants of terminal Gorenstein Fano 3-folds such as their degree or genus are the same as those of non-singular Fano 3-folds, this observation will yield a bound on the rank of the divisor class group of terminal Gorenstein Fano 3-folds that do not contain a plane.

3.1. Minimal Model Program.

Lemma 3.1. *Let X be a small modification of a terminal Gorenstein Fano 3-fold Y . Assume that $|A_X|$ is basepoint free. Denote by $\varphi: X \rightarrow X'$ an extremal divisorial contraction with centre a curve Γ and assume that $\text{Exc } \varphi = E$ is a Cartier divisor.*

Then $\Gamma \subset X'$ is locally a complete intersection and has planar singularities. The contraction φ is locally the blowup of the ideal sheaf \mathcal{I}_Γ . In addition, the following relations hold :

- (1) $(A_X)^3 = (A_{X'})^3 - 2(A_{X'}) \cdot \Gamma - 2 + 2p_a(\Gamma)$
- (2) $(A_X)^2 \cdot E = A_{X'} \cdot \Gamma + 2 - 2p_a(\Gamma)$
- (3) $A_X \cdot E^2 = -2 + 2p_a(\Gamma)$
- (4) $E^3 = -(A_{X'}) \cdot \Gamma + 2 - 2p_a(\Gamma)$

Proof. I only sketch the proof of this Lemma because it is similar to [Cut88]. As the singularities of X are isolated, φ is the blow up of a nonsingular curve Γ away from finitely points of X' , whose fibers contain $\text{Sing } X$. Consider a point $P \in \Gamma$. Let $S \in |A_X|$ be a general section. By Bertini, S is nonsingular and intersects the fiber $\varphi^{-1}(P)$ in a finite set. Since $|A_X| = |\varphi^*A_{X'} - E|$, $S' = \varphi(S)$ is an anticanonical section of X' that contains Γ . The restriction $\varphi|_S: S \rightarrow S'$ is a finite birational map that is an isomorphism outside of $\varphi^{-1}(P)$ and in particular, S' has isolated singularities. Note that S' is Cartier, and since X' is Cohen Macaulay, S' is normal. By Zariski's main theorem, $\varphi|_S$ is an isomorphism. This shows that Γ has planar singularities and that X is the blow up of \mathcal{I}_Γ . The relations then follow from adjunction on X' and from standard manipulation of the equality:

$$A_X = \varphi^*A_{X'} - E.$$

□

Theorem 3.2. *The category of weak-star Fano 3-folds is preserved by the birational operations of the MMP.*

Proof. The category of weak-star Fano 3-folds is clearly stable under flops. Let X be a weak-star Fano 3-fold and $\varphi: X \rightarrow X'$ a divisorial contraction. By Theorem 2.6, X' is \mathbb{Q} -factorial and terminal, and as is explained in Remark 2.2, X' is also Gorenstein.

Step 1. The anticanonical divisor $A_{X'}$ is big because $|A_X| \subset |\varphi^*(A_{X'})|$. I prove that it is also nef. The anticanonical divisors of X and X' satisfy:

$$(5) \quad A_X = \varphi^*(A_{X'}) - aE,$$

where $a = 2$ if φ contracts a plane to a nonsingular point and $a = 1$ otherwise. If φ contracts a divisor to a point, $A_{X'}$ is nef. Assume that φ contracts a divisor E to a curve Γ . Let Z' be an irreducible reduced curve lying on X' . If Z' and Γ intersect in a 0-dimensional set, $A_{X'} \cdot Z' \geq 0$. Thus, $A_{X'}$ can fail to be nef only if $Z' = \Gamma$ and $A_{X'} \cdot \Gamma$ is negative. By relation (2) in Lemma 3.1, this is impossible when X is weak-star Fano because $A_{X'} \cdot \Gamma + 2 \geq A_X^2 \cdot E \geq 2$.

Step 2. Note that A'_X is ample outside of a finite set of curves. Indeed, by (5), if $A_{X'} \cdot C = 0$ for an effective curve C' , then either the proper transform C of C' does not meet the exceptional locus and is such that $A_X \cdot C = 0$ or $C' = \Gamma$ is the centre of φ . As there are finitely many such curves, X' is a small modification of a terminal Gorenstein Fano 3-fold Y' . By Theorem 2.7, the anticanonical degree and Picard

rank of Y' are the same as those that appear in the classifications of [Isk77, Isk78, MM82].

As A_X is basepoint free, (5) shows in addition that $\text{Bs } |A_{X'}|$ is contained in the centre of the contraction φ . If $|A_{X'}|$ is not basepoint free, X' is one of the 3-folds listed in Proposition 2.4. I use the following observations: $A_{X'}^3 > A_X^3$ (see the proof of Lemma 2.3) and if $\rho(Y') > 1$, then $\rho(Y) > 1$ (see the proof of Lemma 3.3). If $\text{Bs } |A_{X'}| = \{p\}$, then $A_{X'}^3 = 2$, [Isk77, Isk78, MM82] show that there is no X with $A_X^3 < 2$. Similarly, it is impossible to find a weak-star X with $\rho(Y) > 1$ and A_X^3 sufficiently small for X' to have $\text{Bs } |A_{X'}| \simeq \mathbb{P}^1$. The linear system $|A_{X'}|$ is basepoint free.

Note in addition, that if A_Y is very ample, so is $A_{Y'}$. Indeed, $R(X', A_{X'})$ is generated in degree 1 if and only if $\Phi_{|A_{X'}|}$ is birational onto its image. Since $\Phi_{|A_X|} = \nu \circ \Phi_{|A_{X'}|}$ where ν is the projection from a possibly empty linear subspace, $A_{Y'}$ is very ample.

Step 3. Let S' be an irreducible divisor on X' and denote by S its proper transform on X . By (5),

$$|A_X|_S = |\varphi^*(A_{X'}) - aE|_S$$

for some $a \in \mathbb{N}$. It is naturally a subsystem of $|A_{X'}|_{S'}$, and the inclusion is strict if S' meets the centre of φ . In particular,

$$(6) \quad (A_X)^2 \cdot S \leq (A_{X'})^2 \cdot S'$$

and X' does not contain an irreducible divisor S' with $(A_{X'})^2 \cdot S' \leq 1$ because X is a weak-star Fano 3-fold. \square

The MMP can be run in the category of weak-star Fano 3-folds. Moreover, there are strong restrictions on the anticanonical models of the intermediate steps.

Lemma 3.3. *Let $X := X_0$ be a weak-star Fano 3-fold whose anticanonical model Y_0 has Picard rank 1. There is a sequence of extremal contractions:*

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\varphi_0} & X_1 & \xrightarrow{\varphi_1} & \cdots & X_{n-1} & \xrightarrow{\varphi_{n-1}} & X_n \\ \downarrow & & \downarrow & & & \downarrow & & \downarrow \\ Y_0 & & Y_1 & & \cdots & Y_{n-1} & & Y_n \end{array}$$

where for each i , X_i is a weak-star Fano 3-fold, Y_i is its anticanonical model, and φ_i is a birational extremal contraction. The Picard rank of Y_i , $\rho(Y_i)$, is equal to 1 for all i . The final 3-fold X_n is either a Fano 3-fold with $\rho(X_n) = 1$ or an extremal Mori fibre space.

Proof. If $\rho(X_i) > 1$, there is an extremal ray R_i that can be contracted. Denote by $\varphi_i: X_i \rightarrow X_{i+1}$ the contraction of R_i . If φ_i is not birational, $X_i = X_n$ is an extremal Mori fibre space and there is nothing to prove.

Assume that φ_i is birational, Theorem 3.2 shows that X_{i+1} is a weak-star Fano 3-fold. Let $h_i: X_i \rightarrow Y_i$ denote its anticanonical map.

I prove that if the Picard rank of Y_i is equal to 1, then the Picard rank of Y_{i+1} is also 1. The statement is trivial when φ_i is a small contraction; assume that φ_i is divisorial and denote by E its exceptional divisor.

The image $h_i(E) = \overline{E} \subset Y_i$ of E by the anticanonical map is a Weil non-Cartier divisor. If h_i was an isomorphism near E , the Cartier divisor \overline{E} would be covered by K_{Y_i} -negative curves and would be contractible in Y_i . This is impossible when $\rho(Y_i) = 1$.

Denote by $f_i: Z_i = \text{Proj} \bigoplus_{n \geq 0} h_{i*} \mathcal{O}_{X_i}(nE) \rightarrow Y_i$ a small partial factorization of Y_i ; Z_i is the symbolic blowup of Y_i along the Weil non-Cartier divisor \overline{E} . Let E' be the Cartier divisor $(h_i)_*^{-1}E$ on Z_i and write $h_i = f_i \circ g_i$. Note that Z_i has Picard rank 2 and that g_i contracts only curves of X_i/Y_i that are disjoint from E .

By the projection formula, E' is covered by K_{Z_i} -negative curves: there is an extremal contraction $\psi_i: Z_i \rightarrow Z_{i+1}$ whose exceptional divisor is E' . Consider the projective and surjective morphism

$$\psi_i \circ g_i: X_i \rightarrow Z_{i+1},$$

and run a relative MMP on X_i over Z_{i+1} . The Contraction Theorem [KMM87] shows that φ_i factorizes $\psi_i \circ g_i$ and makes the diagram

$$\begin{array}{ccc} E \subset X_i & \xrightarrow{\varphi_i} & X_{i+1} \\ g_i \downarrow & & \downarrow g_{i+1} \\ E' \subset Z_i & \xrightarrow{\psi_i} & Z_{i+1} \\ f_i \downarrow & & \\ & & Y_i \end{array}$$

commutative. The morphism g_{i+1} is crepant and X_{i+1} and Z_{i+1} have the same anticanonical model. The Picard rank of Z_{i+1} is equal to 1 and $A_{Z_{i+1}}$ is nef and big: $Y_{i+1} = Z_{i+1}$. \square

Lemma 3.4. *Let X be a weak-star Fano 3-fold and Y its anticanonical model. If $\varphi: X \rightarrow \mathbb{P}^1$ is an extremal del Pezzo fibration of degree k , then $k \geq 2$. If in addition, $R(X, A_X)$ is generated in degree 1, then $k \geq 3$.*

Proof. A general fibre F of φ is a nonsingular del Pezzo surface of degree k . As $A_X|_F = A_F$, the linear system $|A_X|_F$ is naturally a subsystem

of $|A_F|$. Since $|A_X|_F$ is basepoint free, the degree k cannot be equal to 1. If $R(X, A_X)$ is generated in degree 1, the morphism $\Phi_{|A_X|_F}$ is birational onto its image and $k \neq 2$. \square

If $\varphi_n: X_n \rightarrow S$ is a conic bundle, [Cut88] shows that S is nonsingular.

Definition 3.5. A conic bundle $\varphi: X \rightarrow S$ is a *weak Fano conic bundle* (resp. *weak-star*) if X is a weak (resp. weak-star) Fano 3-fold and $\rho(X/S) = 1$.

I recall the following standard result.

Lemma 3.6. [Pro05] *If $\varphi: X \rightarrow S$ is a weak Fano conic bundle, A_S is nef and big.*

Proof. Recall that the discriminant curve Δ of a conic bundle satisfies

$$4A_S = \varphi_*(A_X)^2 + \Delta,$$

where $\varphi_*(A_X)^2$ is nef and big, and therefore A_S is big. Assume that A_S is not nef and let $C \subset S$ be an irreducible curve such that $A_S \cdot C < 0$. The curve C is necessarily contained in Δ , as $\varphi_*(A_X)^2$ is nef and $C^2 \leq C \cdot \Delta < 0$. By adjunction:

$$\begin{aligned} -2 &\leq 2p_a(C) - 2 \leq (A_S + C) \cdot C \\ &\leq (A_S + \Delta) \cdot C \leq (3A_S - \varphi_*(A_X)^2) \cdot C \\ &\leq 3A_S \cdot C. \end{aligned}$$

This is impossible because $A_S \cdot C$ is an integer. \square

Lemma 3.7. *Let $\varphi: X \rightarrow S$ be a weak-star Fano conic bundle. Then there is a weak-star Fano conic bundle $\varphi': X' \rightarrow S'$, with $S' = \mathbb{P}^2, \mathbb{F}_0, \mathbb{F}_1$ or \mathbb{F}_2 , such that the diagram*

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

is commutative.

Proof. If S contains a -1 -curve, denote its contraction by $S \rightarrow S'$. We may run a relative MMP of X over S' and this shows that there is an extremal contraction $X \rightarrow X'$ fitting in a diagram as above. In addition, $X' \rightarrow S'$ has a structure of weak-star conic bundle. There is a sequence of contractions of -1 -curves $S \rightarrow S_1 \rightarrow \cdots \rightarrow S_N$ where S_N is either \mathbb{P}^2 or a \mathbb{P}^1 -bundle over a curve Γ . The surface S_N is clearly rational, so that if it is \mathbb{P}^1 -bundle over a curve Γ , Γ is isomorphic to \mathbb{P}^1 .

In that case, S_N is a Hirzebruch surface $\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$. Since A_{S_N} is nef, S_N is either $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, \mathbb{F}_1 or \mathbb{F}_2 . \square

Theorem 3.8 (End product of the MMP). *Let $X = X_0$ be a weak-star Fano 3-fold whose anticanonical divisor Y has Picard rank 1. The end product of the MMP on X is one of:*

1. X_n is a factorial terminal Fano 3-fold with $\rho(X_n) = 1$.
2. $X_n \rightarrow \mathbb{P}^1$ is an extremal del Pezzo fibration of degree k , with $2 \leq k$ and $\rho(X_n) = 2$.
3. $X_n \rightarrow S$ is a conic bundle over $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_1$ or \mathbb{F}_2 and $\rho(X_n) = 2$ or 3.

3.2. A bound on the defect of some terminal Gorenstein Fano 3-folds.

Notation 3.9. I say that a surface $S \subset Y_{2g-2}$ is a plane (resp. a quadric) if its image by the anticanonical map is a plane (resp. a quadric) in \mathbb{P}^{g+1} , that is when $(A_Y)^2 \cdot S = 1$ (resp. 2).

Lemma 3.10. *Let X be a weak-star Fano 3-fold whose anticanonical model Y does not contain an irreducible quadric surface. Then, every divisorial extremal contraction φ increases the anticanonical degree $(-K)^3 = A^3$ by at least 4.*

Proof. Let $\varphi: X \rightarrow X'$ be a divisorial contraction. I use the notation of Mori-Cutkosky (see Theorem 2.6) for the types of divisorial contractions. The contraction φ is of type E1 or E2. Indeed, φ is not of type E5 because X is weak-star Fano and it is not of type E3 or E4, because Y does not contain an irreducible quadric surface.

If φ is of type E2, $A_X = \varphi^*(A_{X'}) - 2E$ and the degree increases by precisely 8.

If φ is of type E1, I claim that $A_{X'}^3 \geq A_X^3 + 4$. Let E be the exceptional divisor of φ and Γ its centre on X' . As Y does not contain an irreducible quadric surface, $(A_X)^2 \cdot E \geq 3$, and the result follows from Equations (1, 2) in Lemma 3.1. \square

Corollary 3.11. *Let X be a weak-star Fano 3-fold and $\varphi: X \rightarrow X'$ a birational extremal contraction. Denote by Y and Y' the anticanonical models of X and X' . If Y' contains an irreducible quadric surface Q' , then Q' is disjoint from the centre of φ and Y also contains an irreducible quadric surface Q .*

Proof. This is a straightforward consequence of Step 4. of the proof of Theorem 3.2. \square

Corollary 3.12. *Let Y be a terminal Gorenstein Fano 3-fold of index 1, Picard rank 1 and of genus g . If Y does not contain a quadric or a plane, the defect of Y is bounded by $\lfloor \frac{12-g}{2} \rfloor + 4$.*

Proof. Let $h: X \rightarrow Y$ be a factorialization of Y ; X is a weak-star Fano 3-fold. I prove that the Picard rank of X is at most $\lfloor \frac{12-g}{2} \rfloor + 5$.

Run a MMP on X . I follow the notation of Lemma 3.3. The Picard rank of X equals the number of divisorial contractions that occur during the MMP plus the Picard rank of the final 3-fold X_n . Lemma 3.3 shows that at each intermediate step, Y_i has Picard rank 1, and therefore $A_{X_i}^3 = A_{Y_i}^3$ is equal to $2g-2$ with $2 \leq g \leq 10$ or $g = 12$ if $i(X_i) = 1$, $8d$ with $1 \leq d \leq 5$ if $i(X_i) = 2$, 54 if $i(X_i) = 3$ and 64 if $i(X_i) = 4$. Note also that if $X_i \rightarrow X_{i+1}$ is a divisorial contraction, the index of X_i is at most 2 and if $i(X_i) = 2$, $i(X_{i+1})$ is divisible by 2. By lemma 3.10, the number of divisorial contractions is bounded by $\lfloor \frac{12-g}{2} \rfloor + 3$ if $A_{Y_n}^3 \leq 40$ and by $\lfloor \frac{12-g}{2} \rfloor + 4$ otherwise. The Picard rank of X_n is at most 3 and if $i(Y_n) = 4$ (resp. if $i(Y_n) \geq 2$), $\rho(X_n) = 1$ (resp. $\rho(Y_n) \leq 2$) [Shi89]. The bound on the defect follows. \square

Corollary 3.13. *Let Y be a terminal Gorenstein Fano 3-fold of index 1, Picard rank 1 and of genus g . If Y contains a quadric but does not contain a plane, the defect of Y is bounded by $\lfloor \frac{12-n-g}{2} \rfloor + 4 + n$, where $n = \min\{\lfloor \frac{g+1}{3} \rfloor, 10 - g\}$.*

Proof. Let $h: X \rightarrow Y$ be a factorialization of Y ; X is a weak-star Fano 3-fold. I prove that the Picard rank of X is at most $\lfloor \frac{12-n-g}{2} \rfloor + 5 + n$, where $n = \min\{\lfloor \frac{g+1}{3} \rfloor, 10 - g\}$.

Run a MMP on X . If $Q \subset Y$ is an irreducible reduced quadric lying on Y , the index of Y is 1. Denote by \tilde{Q} its proper transform on X . The quadric \tilde{Q} is negative on a K -negative extremal ray R , the class of the proper transform of a ruling of Q , and R may be contracted. Let $\varphi: X \rightarrow X'$ be the contraction of R . If \tilde{Q} is contracted to a point, the anticanonical degree increases by 2. If \tilde{Q} is contracted to a curve Γ , by Lemma 3.1, $A_{X'} \cdot \Gamma = 2p_a(\Gamma)$ and $A_X^3 = A_{X'}^3 - 2(p_a(\Gamma) + 1)$. The contraction φ increases the anticanonical degree by at least 2.

By Lemma 3.10, if Y_i contains a quadric Q_i , Q_i does not intersect the centre of any previous extremal contraction. Denote by \tilde{Q}_i the proper transform of Q_i on X , $h(\tilde{Q}_i)$ is a quadric. We may therefore assume that quadrics are contracted first. While quadrics are contracted, the index of X_i is 1, and as in the proof of Corollary 3.12, the anticanonical degrees $A_{X_i}^3$ can only be of the form $2g-2$, with $2 \leq g \leq 10$ or $g = 12$. There are hence at most $10 - g$ divisorial contractions that decrease the anticanonical degree by 2. Furthermore, the quadrics $h(\tilde{Q}_i) \subset Y \subset$

\mathbb{P}^{g+1} are disjoint, so that at most $\lfloor \frac{g+1}{3} \rfloor$ quadrics may be contracted. If $X_i \rightarrow X_{i+1}$ is the contraction of a quadric, $i(X_i) = i(X_{i+1}) = 1$ and $A_{Y_{i+1}}^3 = 2g - 2$ with $2 \leq g \leq 10$ or $g = 12$. The result then follows from Corollary 3.12. \square

3.3. Higher index Fano 3-folds. Shin classifies canonical Gorenstein Fano 3-folds of index greater than 1 [Shi89]. If Y has terminal Gorenstein singularities and index 2, 3 or 4, it has Picard rank 1 and if $X \rightarrow Y$ is a small factorialization, X is a weak-star Fano 3-fold.

Lemma 3.14. *Let X be an index 2 weak-star Fano 3-fold and let $\varphi: X \rightarrow X'$ be the contraction of an extremal ray. Then, one of the following holds:*

1. φ is birational, and φ is either a flop or an E2 contraction.
2. $\varphi: X \rightarrow S$ is an étale \mathbb{P}^1 -bundle.
3. $\varphi: X \rightarrow \mathbb{P}^1$ is a del Pezzo fibration of degree 8, and φ is a quadric bundle.

Remark 3.15. Similarly, if X has index 3 and $\rho(X) \geq 2$, then $\rho(X) = 2$. Any contraction of an extremal ray is either a flop or a del Pezzo fibration of degree 9, that is, X is a \mathbb{P}^2 -bundle over \mathbb{P}^1 .

Corollary 3.16. *Let Y be a terminal Gorenstein Fano 3-fold of index 2 and X a small factorialization of Y . Denote by $h^3 = \frac{A_X^3}{8}$ the degree of X and Y . The Picard rank of X is at most $8 - h^3$. The 3-fold X contains at most $7 - h^3$ disjoint planes.*

Proof. Each divisorial contraction increases the degree by 1: this proves the first assertion.

Let $E = \mathbb{P}^2$ be a plane contained in X , then $A_{X|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(2)$. The contraction theorem shows that there is an extremal contraction that contracts E to a point.

For a suitable choice of X , all birational contractions appearing in the MMP are divisorial. Indeed, a flopping contraction and a divisorial contraction of type E2 always commute [CJR08], therefore we may assume that flops are performed first. The proper transforms on X of the divisors contracted when running the MMP on X are disjoint planes. There is no other plane disjoint from these lying on X , as this would give rise to another extremal contraction. \square

4. INDEX 1 FANO 3-FOLDS THAT CONTAIN A PLANE

This section is independent of the rest of the paper. I consider terminal Gorenstein Fano 3-folds whose small factorializations are not weak-star Fanos: the rank of their divisor class group cannot be bounded with

the techniques developed in the preceding sections. I first bound the rank of the divisor class group of quartic hypersurfaces by explicit calculation and then indicate how to treat other Fano 3-folds with Picard rank 1.

4.1. $Y_4 \subset \mathbb{P}^4$, **genus 3**. Let $Y = Y_4^3 \subset \mathbb{P}^4$ be a quartic 3-fold with terminal singularities that contains a plane $\Pi = \{x_0 = x_1 = 0\}$. As Y is a hypersurface in \mathbb{P}^4 , it is Gorenstein and has Picard rank 1. The equation of Y is of the form:

$$(7) \quad Y = \{x_0 a_3(x_0, \dots, x_4) + x_1 b_3(x_0, \dots, x_4) = 0\}.$$

Let X be the blowup of Y along the plane Π . Locally, the equation of X can be written:

$$(8) \quad \{t_0 a_3(t_0 x, t_1 x, x_2, x_3, x_4) + t_1 b_3(t_0 x, t_1 x, x_2, x_3, x_4) = 0\} \\ \subset \mathbb{P}(t_0, t_1) \times \mathbb{P}(x, x_2, x_3, x_4),$$

where the variable x is defined by $x_0 = t_0 x$, $x_1 = t_1 x$.

The 3-fold X has a natural structure of cubic del Pezzo fibration over \mathbb{P}^1 . The generic fibre X_η is reduced and irreducible. However, special fibres might be reducible. The cubic fibration X is a small partial factorization of Y , in particular the ranks of $\text{Cl } X$ and of $\text{Cl } Y$ are equal.

Lemma 4.1. *Denote by N the number of reducible fibres with 3 irreducible components and by M the number of reducible fibres with 2 irreducible components. The rank of $\text{Cl } X$ is bounded by $8 + 2N + M$.*

Proof. Let \mathcal{O} be the local ring of \mathbb{P}^1 at a point P corresponding to a reducible fibre of X . Let K be its fraction field, t its parameter and $k = \mathbb{C}$ its residue field. Let $S = \text{Spec}(\mathcal{O})$, $\eta = \text{Spec } K$ the generic point and let $0 = \text{Spec } k$ be the origin. Let $\mathbb{P} = \mathbb{P}_{\mathcal{O}}^n$ be an n -dimensional projective space over S and $L = L_d$ a d -dimensional projective subspace, defined over k . If $d \leq n - 1$, there is a birational transformation of \mathbb{P} centred at L :

$$(x_0 : \dots : x_n) \longmapsto (tx_0 : \dots : tx_d : x_{d+1} : \dots : x_n) ;$$

ϕ_L is the projection from L .

Corti shows [Cor96, Flowchart 2.13, Lemma 2.17 and Corollary 2.20] that, if $X_{\mathcal{O}} \subset \mathbb{P}_{\mathcal{O}}^3$ is a weak Fano cubic fibration with small anti-canonical map, projecting away from planes contained in the central fibre yields a standard integral model $X'_{\mathcal{O}}$ for X_η , i.e. a flat subscheme $X'_{\mathcal{O}} \subset \mathbb{P}_{\mathcal{O}}^3$ with isolated cDV singularities and reduced and irreducible central fibre. Each projection from a plane contained in the central

fibre only affects the cubic fibration in the central fibre and it strictly decreases the number of k -irreducible components of the central fibre.

If a fibre of $X \rightarrow \mathbb{P}^1$ is reducible, it has at most 3 irreducible components (one of which at least is a projective plane). Project away from planes in reducible fibres of X in order to obtain a standard integral model. The lemma then follows from the sequence:

$$0 \rightarrow \mathbb{Z}[\pi^* \mathcal{O}_{\mathbb{P}^1}(1)] \rightarrow \text{Weil}(X') \rightarrow \text{Pic}(X'_\eta) \rightarrow 0,$$

which is exact when $\pi: X' \rightarrow \mathbb{P}^1$ has reduced and irreducible fibres: \square

Bounding the rank of $\text{Cl } Y$ is therefore equivalent to bounding the number of reducible fibres of X .

Theorem 4.2. *Let $Y = Y_4^3 \subset \mathbb{P}^4$ be a terminal quartic 3-fold that contains a plane Π and let X be the cubic del Pezzo fibration obtained by blowing up Y along Π . The cubic fibration X has at most 4 reducible fibres : the rank of $\text{Cl } Y$ is at most 16.*

Proof. Let

$$H_{(\lambda:\mu)} = \{\lambda x_0 + \mu x_1 = 0\} \subset \mathbb{P}^4, (\lambda:\mu) \in \mathbb{P}^1$$

be the pencil of hyperplanes of \mathbb{P}^4 that contain Π .

The hyperplane section of $Y \subset \mathbb{P}^4$ corresponding to $H_{(\lambda:\mu)}$ is a reducible quartic surface that contains the plane Π ; more precisely $Y \cap H_{(\lambda:\mu)} = \Pi \cup Y'_{(\lambda:\mu)}$, where $Y'_{(\lambda:\mu)}$ is a possibly reducible cubic surface, naturally isomorphic to the fibre $X_{(\lambda:\mu)}$.

If X has a reducible fibre over $(\lambda:\mu) \in \mathbb{P}^1$, X contains a plane either of the form:

$$\Pi_{(\lambda:\mu)} = \{\lambda t_0 + \mu t_1 = l(x_2, x_3, x_4) + l'(t_0 x, t_1 x) = 0\},$$

where l and l' are linear, or of the form:

$$\Pi_{(\lambda:\mu)} = \{\lambda t_0 + \mu t_1 = x = 0\}.$$

The plane $\Pi_{(\lambda:\mu)} \subset X_{(\lambda:\mu)}$ corresponds to a plane $\Pi'_{(\lambda:\mu)}$ lying on $Y \cap H_{(\lambda:\mu)}$. In the first case, $\Pi'_{(\lambda:\mu)}$ is of the form:

$$\Pi'_{(\lambda:\mu)} = \{\lambda x_0 + \mu x_1 = l(x_2, x_3, x_4) + l'(x_0, x_1) = 0\}$$

and meets Π in a line $L_{(\lambda:\mu)} = \{x_0 = x_1 = l(x_2, x_3, x_4) = 0\}$, while in the second case $\Pi'_{(\lambda:\mu)} = \Pi$ and $H_{(\lambda:\mu)}$ is tangent to Y along Π .

Step 1. I first show that no hyperplane is tangent to Y along Π .

If this is the case, then without loss of generality, we may assume that the hyperplane $H_{(1:0)}$ is tangent to Y along Π . The equation of Y then reads

$$\{x_0 a_3 + x_1 x_0 a_2 + x_1^2 b_2 = 0\}$$

where a_3 is a homogeneous form of degree 3 and a_2 and b_2 are homogeneous forms of degree 2. The quartic Y is singular along the curve $\Gamma = \{a_3 = x_0 = x_1 = 0\}$: this contradicts Y having terminal singularities.

Step 2. I show that if Π'_1 and Π'_2 are two planes distinct from Π that lie on distinct hyperplane sections of Y , then they intersect Π in distinct lines.

Let $L_1 = \Pi \cap \Pi'_1$ and $L_2 = \Pi \cap \Pi'_2$. If $L_1 = L_2$, up to coordinate change on \mathbb{P}^4 , I may assume that $\Pi'_1 = \{x_0 = x_2 = 0\}$ and $\Pi'_2 = \{x_1 = x_2 = 0\}$. The plane Π'_1 (resp. Π'_2) lies on Y if and only if, in (7), the homogeneous form b_3 is in the ideal $\langle x_0, x_2 \rangle$ of Π'_1 (resp. the homogeneous form a_3 is in the ideal $\langle x_1, x_2 \rangle$ of Π'_2). The quartic Y then has multiplicity 2 along the line $L = \{x_0 = x_1 = x_2 = 0\}$. This is a contradiction: Π'_1 and Π'_2 meet at a point.

Step 3. I now show that if Π'_1, Π'_2 and Π'_3 are 3 planes that lie on distinct hyperplane sections of Y and are distinct from Π , the lines $L_1 = \Pi \cap \Pi'_1, L_2 = \Pi \cap \Pi'_2$ and $L_3 = \Pi \cap \Pi'_3$ are not concurrent.

Assume that the planes Π'_1, Π'_2 and Π'_3 meet at a point P . Up to coordinate change on \mathbb{P}^4 , we may assume that:

$$\begin{aligned}\Pi'_1 &= \{x_0 = x_2 = 0\} \\ \Pi'_2 &= \{x_1 = x_3 = 0\} \\ \Pi'_3 &= \{x_0 + x_1 = x_2 + x_3 + l(x_0, x_1) = 0\},\end{aligned}$$

where l is a linear form. The equation of Y is in the ideal spanned by the monomials:

$$\begin{aligned}I = \langle & (x_0 + x_1)x_0x_1, x_0x_1(x_2 + x_3 + l(x_0, x_1)), (x_0 + x_1)x_1x_2, (x_0 + x_1)x_0x_3, \\ & x_1x_2(x_2 + x_3 + l(x_0, x_1)), x_0x_3(x_2 + x_3 + l(x_0, x_1)), (x_0 + x_1)x_2x_3, \\ & x_2x_3(x_2 + x_3 + l(x_0, x_1)) \rangle,\end{aligned}$$

hence $P = (0:0:0:0:1) = L_1 \cap L_2 \cap L_3 \in Y$ is not a cDV point. If X contains at least three distinct reducible fibres, then up to coordinate change on \mathbb{P}^4 , we may assume that

$$\begin{aligned}\Pi'_1 &= \{x_0 = x_2 = 0\} \subset H_{(0:1)} \\ \Pi'_2 &= \{x_1 = x_3 = 0\} \subset H_{(1:0)} \\ \Pi'_3 &= \{x_0 + x_1 = x_4 = 0\} \subset H_{(1:1)}.\end{aligned}$$

lie on Y .

Step 4. Finally, I prove that X has at most 4 reducible fibres.

Assume that X has at least 5 reducible fibres and let $\Pi'_1, \dots, \Pi'_5 \subset \mathbb{P}^4$ be planes that are distinct from Π and that lie on distinct hyperplane sections of Y . Denote by L_i the line $\Pi \cap \Pi'_i$ for $1 \leq i \leq 5$. Steps 2 and

3 show that the lines $L_i \subset \Pi$ are distinct and that no three of these lines are concurrent.

We may therefore assume that:

$$\begin{aligned}\Pi'_1 &= \{x_0=x_2=0\} \\ \Pi'_2 &= \{x_1=x_3=0\} \\ \Pi'_3 &= \{x_0+x_1=x_4=0\} \\ \Pi'_4 &= \{x_0+\lambda x_1=ax_2+bx_3+cx_4+l(x_0,x_1)=0\}\end{aligned}$$

where a, b and c are constants and l is a linear form. The constant $\lambda \neq 0, 1$ or ∞ because no two of the planes Π_i lie in the same hyperplane section of Y . The constants a, b and c are all non-zero, as otherwise either two of the lines L_i coincide, or three of the lines L_i meet at a point. Up to rescaling, we may assume that:

$$\Pi'_4 = \{t_0 + \lambda t_1 = x_2 + x_3 + x_4 + l(t_0, t_1)x = 0\}.$$

The equation of Π'_5 is of the form:

$$\Pi'_5 = \{x_0 + \mu x_1 = \alpha x_2 + \beta x_3 + \gamma x_4 + l'(x_0, x_1) = 0\},$$

where l' is a linear form and α, β and γ are constants. As any three lines of L_1, \dots, L_5 have to satisfy the conditions of Steps 2 and 3, α, β and γ are all non-zero. I may assume that $\alpha = 1$. Considering triples of lines (L_4, L_5, L_i) for $1 \leq i \leq 3$ shows that $\beta \neq 1$, $\gamma \neq 1$, and $(\beta:\gamma) \neq (1:1)$. The equation of Y may be written uniquely in the form:

$$(9) \quad f(x_0, \dots, x_4) + g(x_0, \dots, x_4) = 0,$$

where the monomials that appear in f have degree at least 2 in x_0, x_1 , while $g(x_0, \dots, x_4) = x_0 g_0(x_2, x_3, x_4) + x_1 g_1(x_2, x_3, x_4)$ has degree exactly 1 in x_0, x_1 . In the expression (9), g may be written as a linear combination of the monomials:

$$\begin{aligned}\langle x_0 x_3 x_4(x_2 + x_3 + x_4), x_1 x_2 x_4(x_2 + x_3 + x_4), \\ (x_0 + x_1)x_2 x_3(x_2 + x_3 + x_4), (x_0 + \lambda x_1)x_2 x_3 x_4 \rangle.\end{aligned}$$

or as a linear combination of the monomials:

$$\begin{aligned}\langle x_0 x_3 x_4(x_2 + \beta x_3 + \gamma x_4), x_1 x_2 x_4(x_2 + \beta x_3 + \gamma x_4), \\ (x_0 + x_1)x_2 x_3(x_2 + \beta x_3 + \gamma x_4), (x_0 + \mu x_1)x_2 x_3 x_4 \rangle.\end{aligned}$$

Equating these two expressions shows that either $g = 0$ or $\lambda = \mu$. If $g = 0$, Y does not have isolated singularities; If $\lambda = \mu$, Π'_4 and Π'_5 are both contained in the hyperplane section $H_{(1:\lambda)}$. Both cases yield a contradiction: the cubic fibration X has at most 4 reducible fibres. \square

Example 4.3 (The Burkhardt quartic). It is known that a nodal quartic hypersurface in \mathbb{P}^4 has at most 45 nodes [Var83, Fri86]. Up to projective equivalence, there is only one 45-nodal quartic 3-fold $Y \subset \mathbb{P}^4$ [dJSBVdV90] and its equation is:

$$\{x_0^4 - x_0(x_1^3 + x_2^3 + x_3^3 + x_4^3) + 3x_1x_2x_3x_4 = 0\}.$$

Let \tilde{Y} be the blowup of Y at the 45 nodes. The defect of Y satisfies $\sigma(Y) = b_2(\tilde{Y}) - 45 - b_2(Y) = b_2(\tilde{Y}) - 46$. The cohomology of \tilde{Y} is determined in [HW01] and $b_2(\tilde{Y}) = 61$, so that the defect of the Burkhardt quartic is 15.

Alternatively, the plane $\Pi = \{x_0 = x_1 = 0\}$ is contained in Y . Let X be the 3-fold obtained by blowing up Y along the plane Π ; the equation of X is:

$$t_0(t_0^3 - t_1^3)x^3 - t_0(x_2^3 + x_3^3 + x_4^3) + 3t_1x_2x_3x_4 = 0,$$

and on the affine piece $t_1 \neq 0$, this reads

$$t_0(t_0^3 - 1)x^3 - t_0(x_2^3 + x_3^3 + x_4^3) + 3x_2x_3x_4 = 0.$$

The central fibre, which corresponds to $(t_0:t_1) = (0:1)$, has three irreducible components: the planes $\{t_0 = x_i = 0\}$. Each fibre is reduced and irreducible for $(t_0:t_1) \neq (0:1)$ and $(t_0:t_1) \neq (1:\omega^i)$ for $0 \leq i \leq 2$, where ω is a cube root of unity (notice that considering the affine piece $t \neq 1$ yields the same results). The generic fibre X_η is a nonsingular cubic surface in \mathbb{P}^3 .

Consider for instance the fibre X_1 over $(1:1)$. The fibre X_1 is the union of three planes in $\mathbb{P}(x, x_2, x_3, x_4)$: $\Pi_1 = \{x_2 + x_3 + x_4 = 0\}$, $\Pi_2 = \{\omega x_2 + x_3 + \omega^2 x_4 = 0\}$ and $\Pi_3 = \{\omega^2 x_2 + x_3 + \omega x_4 = 0\}$. The situation is analogous for the other two reducible fibres. There are 27 closed subschemes in $X \setminus \bigcup_{i=0,1,2} X_i$ isomorphic to $\mathbb{P}_{\mathbb{P}^1 \setminus (\cup_i \{(\omega^i:1)\})}^1$. In other words, the 27 lines on the generic fibre may be completed to divisors on X , they are rational over $\mathbb{P}^1 \setminus (\cup_i \{(\omega^i:1)\})$. The Picard rank of X_η is 7 and the generators of $\text{Pic}(X_\eta)$ complete to independent Cartier divisors on X .

The rank of the Weil group of the Burkhardt quartic is 16 and its defect is 15.

Corollary 4.4. *Let $Y_4 \subset \mathbb{P}^4$ be a quartic 3-fold with defect 15. Then Y is projectively equivalent to the Burkhardt quartic.*

Proof. The quartic $Y_4 \subset \mathbb{P}^4$ has defect 15, hence by Corollary 3.13, Y contains a plane $\Pi = \{x_0 = x_1 = 0\}$. Denote by X the blowup of Y along the plane Π . By Lemma 4.1, the cubic fibration X has exactly 4 reducible fibres, each of which has exactly three irreducible components. As in the proof of Theorem 4.2, we use the description of planes

contained in reducible fibres of X to carry a refined analysis of the monomials appearing in the equation of X and Y . If the equation of Y reads

$$\{x_0a_3 + x_1b_3 = 0\},$$

where a_3, b_3 are homogeneous cubic polynomials, and if X has 4 reducible fibres, it is easy to see that $\{x_0 = x_1 = a_3 = b_3 = 0\}$ consists of exactly 9 points, which hence have to be ordinary double points. There are 9 flopping lines in X that lie above these 9 ordinary double points. The cubic fibration X is nonsingular outside of its 4 reducible fibres. Using calculations analogous to those in the proof of Theorem 4.2, there are precisely 9 singular points lying on each reducible fibre, and these therefore are ordinary double points. The quartic Y is nodal and has at least 45 ordinary double points. By [Var83, dJSBVdV90], Y is projectively equivalent to the Burkhardt quartic. \square

4.2. Higher genera. The genus g of Y satisfies $3 \leq g \leq 10$ or $g = 12$ [Muk02]. Mukai shows that if $g = 12$, Y is nonsingular; Y is factorial and therefore does not contain a plane. If $g = 10$, $Y_{18} \subset \mathbb{P}^{11}$ is a linear section of a 5-dimensional G_2 -variety in \mathbb{P}^{13} . If $g = 9$, $Y_{16} \subset \mathbb{P}^{10}$ is a linear section of the 6-dimensional symplectic grassmannian $G \subset \mathbb{P}^{12}$. In either case, Y does not contain a plane, because neither $G_2 \subset \mathbb{P}^{13}$ nor $G \subset \mathbb{P}^{12}$ does [LM03].

Let $Y_{2g-2} \subset \mathbb{P}^{g+1}$ for $4 \leq g \leq 8$ be a Fano 3-fold with terminal Gorenstein singularities that contains a plane $\Pi \simeq \mathbb{P}^2$. The defect of Y can be bounded by studying the projection from the plane $\Pi \subset Y$. Let $\mu: X \rightarrow Y$ be the blowup of Y along Π , so that the projection from Π decomposes as:

$$\begin{array}{ccc} & X & \\ \mu \swarrow & & \searrow \pi \\ Y & \overset{\Phi_\Pi}{\dashrightarrow} & V. \end{array}$$

1. $Y_{2,3} \subset \mathbb{P}^5, g = 4$. The morphism π is K_X -negative and is a conic bundle over $V = \mathbb{P}^2$. The rank of the Weil group of X is bounded above by $1 + \deg(\Delta)$, where Δ is the discriminant curve of π in \mathbb{P}^2 . The 3-fold X is a complete intersection of sections of the linear systems $|M + L|$ and $|2M + L|$ on the scroll $\mathbb{F}_{0,0,1}$ over \mathbb{P}^2 . Computations on the scroll show that $\deg(\Delta) \leq 7$, so that the defect of Y is bounded by 8.
2. $Y_{2,2,2} \subset \mathbb{P}^6, g = 5$. The morphism π is birational, maps onto $V = \mathbb{P}^3$ and A_X is π -ample. The 3-fold X is a complete intersection of members of $|M + L|$ on the scroll $\mathbb{F}_{0,0,1}$ over \mathbb{P}^3 .

In order to bound the defect of Y , it is enough to bound the numbers of points of \mathbb{P}^3 such that the three sections of $|M + L|$ in the fibre of the scroll $\mathbb{F}_{0,0,1}$ are not linearly independent. Indeed, the centre of the morphism π consists of a finite number of points because Y has isolated singularities. Crude considerations on the birational map $X \rightarrow \mathbb{P}^3$ show that the defect of Y is bounded above by 7.

3. $g = 6, 7, 8$. The morphism π is birational, A_X is π -ample, and maps X onto V_d , a Fano 3-fold of index 2 and degree $d = 3$ when $g = 6$, 4 when $g = 7$, and 5 when $g = 8$. The 3-fold V_d has isolated canonical singularities. We can adapt some of the methods in [Tak06], and use the explicit descriptions of linear subspaces of Y that follow from [Muk02] to study the birational map $X \rightarrow Y$. The defect of Y is bounded above by 5 when $g = 6, 7$ or $g = 8$.

Proposition 4.5.

1. *The defect of a terminal Gorenstein $Y_{2,3} \subset \mathbb{P}^5$ that contains a plane is at most 8,*
2. *The defect of a terminal Gorenstein $Y_{2,2,2} \subset \mathbb{P}^6$ that contains a plane is at most 7,*
3. *The defect of a Picard rank 1 terminal Gorenstein Fano 3-fold of genus $g = 6, 7$ or 8 that contains a plane is at most 5.*

4.3. Non anticanonically embedded Fano 3-folds. I say a few words about the Picard rank 1 terminal Gorenstein Fano 3-folds whose anticanonical ring are not generated in degree 1 and that contain a plane. First recall from the proof of Lemma 2.3 that the case of the double quadric does not occur, so that I only need to consider the case $g = 2$. It is known that the defect of a double sextic with no worse than ordinary double points is at most 13 and that this bound is attained by a double cover of \mathbb{P}^3 ramified along the Barth sextic surface [Wah98, End99]. I conjecture that this bound holds for terminal Gorenstein singularities, although I have little evidence to support this.

Conjecture 4.6. *The defect of a double sextic with terminal Gorenstein singularities is at most 13.*

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